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Dualisation of the $D = 9$ matter coupled supergravity

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ABSTRACT: We perform the bosonic dualisation of the matter coupled $\mathcal{N} = 1$, $D = 9$ supergravity. We derive the Lie superalgebra which parameterizes the coset map whose Cartan form realizes the second-order bosonic field equations. Following the non-linear coset construction we present the first-order formulation of the bosonic field equations as a twisted self-duality condition.

KEYWORDS: Global Symmetries, Superstrings and Heterotic Strings, Supersymmetry and Duality, Supergravity Models.

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Contents

1. Introduction

The dualisation of the bosonic sectors of the IIB supergravity [1−3] and the maximal supergravities which are obtained from the $D = 11$ supergravity [4] by torodial compactifications is performed in [5]. The bosonic dualisation which is another manifestation of the Lagrange multiplier method [6] has enabled us to construct the bosonic sectors of the above mentioned supergravity theories as non-linear realisations. This can be considered as a generalization of the non-linear structure of the scalar cosets of these supergravity theories to include the bosonic fields as well. The method of [5] has also served to obtain the first-order formulation of the bosonic sectors in which the first-order field equations arise from a twisted self-duality condition which the dualized Cartan form obeys. The scalar coset manifolds of the supergravities studied in [5] have split real form global symmetry groups. The dualisation of a generic scalar coset which has a split real form global symmetry group is performed in [7]. In [8] this formulation is generalized to the non-split scalar cosets [9, 10]. The dualisation and the first-order formulation of the general non-split scalar coset which is coupled to generic m-form matter fields is performed in [11]. In the light of [8] and [11] the dualisation method of [5] has been extended to cover the class of supergravities which possess scalar cosets based on non-split global symmetry groups and which contain matter couplings. The dualisation and the non-linear realisation of the $D = 8$ and the $D = 7$ matter coupled supergravities are studied in [12] and [13] respectively.

In this work we present the non-linear realisation of the $\mathcal{N} = 1$, $D = 9$ supergravity which is coupled to N vector multiplets [14]. In section two we will introduce the matter coupled $D = 9$ supergravity and we will discuss the scalar coset manifold. We will give the construction of the scalar lagrangian as a symmetric space sigma model [5−10, 15]. The second-order bosonic field equations will also be derived in section two. In section three we will derive the Lie superalgebra which parameterizes the doubled coset element that generates the dualized Cartan form. The construction will be based on the requirement that the dualized Cartan form should yield the second-order bosonic field equations obtained in section two via the Cartan-Maurer equation. Finally we will explicitly calculate the dualized Cartan form and obtain the first-order field equations by using the twisted self-duality condition which the dualized Cartan form satisfies [5].

2. The $D = 9$ matter coupled supergravity

The $\mathcal{N} = 1$, $D = 9$ supergravity is coupled to N vector multiplets in [14]. The $\mathcal{N} = 1$, $D = 9$ supergravity multiplet can be given as

$$(e^r_\mu, \psi_\mu, \chi, B_{\mu\nu}, A_\mu, \sigma)\,, \tag{2.1}$$

where e^r_μ is the veilbein, $B_{\mu\nu}$ is a two-form field, A_μ is a one-form field, σ is the dilaton whereas ψ_μ and χ are the spinor fields. The matter coupled $D = 9$ supergravity can be obtained by coupling (2.1) an arbitrary number N of $\mathcal{N} = 1$ vector multiplets

$$(\lambda, A_\mu, \varphi)\,, \tag{2.2}$$

where λ is a spinor field, A_μ is a one-form and φ is the scalar. Thus the total field content of the $D = 9$ matter coupled supergravity becomes

$$(e^r_\mu, \psi_\mu, \chi, B_{\mu\nu}, A^I_\mu, \sigma, \lambda^\alpha, \varphi^\alpha)\,, \tag{2.3}$$

where $\alpha = 1, \ldots, N$ and $I = 1, \ldots, N + 1$. Our dualisation analysis in the next section will be applied to the bosonic sector of this theory therefore we are not interested in the nature of the spinor fields.

The scalars φ^α for $\alpha = 1,\ldots, N$ parameterize the scalar coset manifold $\mathrm{SO}(N, 1)/\mathrm{SO}(N)$ where $\mathrm{SO}(N, 1)$ is in general a non-compact real form of a semi-simple Lie group and $\mathrm{SO}(N)$ is its maximal compact subgroup. For this reason $\mathrm{SO}(N, 1)/\mathrm{SO}(N)$ is a Riemannian globally symmetric space for all the $\mathrm{SO}(N, 1)$-invariant Riemannian structures on $\mathrm{SO}(N, 1)/\mathrm{SO}(N)$ [16]. Therefore the scalar sector which consists of the vector multiplet scalars of the $D = 9$ matter coupled supergravity can be formulated as a general symmetric space sigma model [8]. To construct the symmetric space sigma model lagrangian we will use the solvable Lie algebra parametrization [17] for the parametrization of the scalar coset manifold $\mathrm{SO}(N, 1)/\mathrm{SO}(N)$. The solvable Lie algebra parametrization is a consequence of the Iwasawa decomposition [16]

$$\begin{aligned} so(N, 1) &= \mathbf{k}_0 \oplus \mathbf{s}_0 \\ &= \mathbf{k}_0 \oplus \mathbf{h_k} \oplus \mathbf{n_k}\,, \end{aligned} \tag{2.4}$$

where $\mathbf{k}_0$ is the Lie algebra of $\mathrm{SO}(N)$ and $\mathbf{s}_0$ is the solvable Lie subalgebra of $so(N, 1)$. $\mathbf{h_k}$ is a subalgebra of the Cartan subalgebra $\mathbf{h}_0$ of $so(N, 1)$ which generates the maximal R-split torus in $\mathrm{SO}(N, 1)$ [8, 10, 16]. The nilpotent Lie subalgebra $\mathbf{n_k}$ of $so(N, 1)$ is generated by a subset $\{E_m\}$ of the positive root generators of $so(N, 1)$ where $m \in \Delta^+_{nc}$. The roots in Δ^+_{nc} are the non-compact roots with respect to the Cartan involution θ induced by the Cartan decomposition [8, 11, 16]

$$so(N, 1) = \mathbf{k}_0 \oplus \mathbf{u}_0\,, \tag{2.5}$$

where $\mathbf{u}_0$ is a vector subspace of $so(N, 1)$. By using the scalar fields φ^α of the coupling vector multiplet and the generators of the solvable Lie algebra $\mathbf{s}_0$ we can parameterize the representatives of the scalar coset manifold $\mathrm{SO}(N, 1)/\mathrm{SO}(N)$ as [16]

$$L = \mathbf{g}_H \mathbf{g}_N = e^{\frac{1}{2}\phi^i H_i} e^{\chi^m E_m}\,, \tag{2.6}$$

where $\{H_i\}$ for $i = 1, \ldots, \dim \mathbf{h}_k$ are the generators of $\mathbf{h}_k$ and $\{E_m\}$ for $m \in \Delta_{nc}^{+}$ are the positive root generators which generate $\mathbf{n_k}$. The scalars $\{\phi^i\}$ for $i = 1, \ldots, \dim \mathbf{h}_k$ are called the dilatons and $\{\chi^m\}$ for $m \in \Delta_{nc}^{+}$ are called the axions. The coset representatives satisfy the defining equations of $\mathrm{SO}(N,1)$

$$L^T \eta L = \eta \,, \qquad L^{-1} = \eta L^T \eta \,, \tag{2.7}$$

where $\eta = \mathrm{diag}(-,+,+,+,\ldots,+)$. Likewise in [12] and [13] we will assume an $(N+1)$-dimensional representation for the algebra $so(N,1)$ which generates symmetric matrix representatives for the coset elements of (2.6). Therefore $L^T = L$ but as we have done in [12] and [13] we will keep on using L^T in our formulation for the sake of accuracy. By using the definitions (2.6) and (2.7), also by using the fact that we have chosen the $(N+1)$-dimensional representation for the algebra $so(N,1)$ such that $L^T = L$ we have the following relations for the coset representatives under the representation chosen

$$\begin{aligned}
\partial_i L \equiv \frac{\partial L}{\partial \phi^i} &= \frac{1}{2} H_i L \,, \qquad \partial_i L^T \equiv \frac{\partial L^T}{\partial \phi^i} = \frac{1}{2} L^T H_i^T \,, \\
\partial_i L^{-1} \equiv \frac{\partial L^{-1}}{\partial \phi^i} &= -\frac{1}{2} L^{-1} H_i \,, \\
(H_i L)^T &= H_i L \,, \qquad\qquad\qquad H_i^T \eta = -\eta H_i \,.
\end{aligned} \tag{2.8}$$

Now we will introduce the internal metric $\mathcal{M}$ as

$$\mathcal{M} = L^T L \,. \tag{2.9}$$

The scalar lagrangian which governs the N scalar fields of the vector multiplet can be constructed as

$$\mathcal{L}_{scalar} = \frac{1}{16} \operatorname{tr} \left(* d\mathcal{M}^{-1} \wedge d\mathcal{M} \right) \,, \tag{2.10}$$

which is the lagrangian for the symmetric space sigma model [5, 8, 10, 11]. Apart from gravity the bosonic lagrangian of the $D = 9$ matter coupled supergravity can now be given as [14]

$$\mathcal{L} = \frac{7}{4} * d\sigma \wedge d\sigma + \frac{1}{2} e^{-4\sigma} * G \wedge G + \frac{1}{16} \operatorname{tr}(* d\mathcal{M}^{-1} \wedge d\mathcal{M}) - \frac{1}{2} e^{-2\sigma} F \wedge \mathcal{M} * F \,, \tag{2.11}$$

where the coupling between the field strengths $F^I = dA^I$ for $I = 1, \ldots, (N+1)$ and the scalars which parameterize the coset $\mathrm{SO}(N,1)/\,\mathrm{SO}(N)$ can be explicitly written as

$$-\frac{1}{2} e^{-2\sigma} F \wedge \mathcal{M} * F = -\frac{1}{2} e^{-2\sigma} \mathcal{M}_{IJ} F^I \wedge * F^J \,. \tag{2.12}$$

The Chern-Simons three-form G is defined as [14]

$$G = dB + \eta_{IJ} A^I \wedge F^J \,. \tag{2.13}$$

We should remark one point here that the construction of [14] is built on a symmetric invariant second-rank tensor L_{IJ} of the group $\mathrm{SO}(N,1)$ which is specified by the choice of the $(N+1)$-dimensional representation for the coset elements L of $\mathrm{SO}(N,1)/\mathrm{SO}(N)$. In our formulation we have used the solvable Lie algebra parametrization (2.6) whose images are in the matrix group $\mathrm{SO}(N,1)$ as a result of the exponential map and the proposed fundamental representation of $so(N,1)$. Thus in our assumed representation our coset representatives are elements of the global symmetry group $\mathrm{SO}(N,1)$. Therefore due to the definition of L_{IJ} in [14] $L_{IJ}=\eta_{IJ}$ for our general coset representation. This is in conformity with the construction in [12] and [13].

By varying the lagrangian (2.11) with respect to the fields σ, B and $\{A^I\}$ we find the corresponding field equations as

$$\begin{aligned}\frac{7}{2}d(*d\sigma) &= -2e^{-4\sigma}*G\wedge G+e^{-2\sigma}\mathcal{M}_{IJ}*F^I\wedge F^J\,,\\ d(e^{-4\sigma}*G) &= 0\,,\\ d(e^{-2\sigma}\mathcal{M}_{IJ}*F^J) &= 2e^{-4\sigma}\eta_{IJ}F^J\wedge *G\,.\end{aligned} \tag{2.14}$$

In [8] the Cartan-Maurer form

$$\mathcal{G}_0 = dLL^{-1}\,, \tag{2.15}$$

for a generic scalar coset is already calculated as

$$\begin{aligned}\mathcal{G}_0 &= \frac{1}{2}d\phi^i H_i + e^{\frac{1}{2}\alpha_i\phi^i}U^\alpha E_\alpha\\ &= \frac{1}{2}d\phi^i H_i + \vec{\mathbf{E}}'\,\mathbf{\Omega}\,\vec{d\chi}\,.\end{aligned} \tag{2.16}$$

Here we have introduced the column vector

$$U^\alpha = \mathbf{\Omega}^\alpha_\beta d\chi^\beta\,, \tag{2.17}$$

and the row vector

$$(\vec{\mathbf{E}}')_\alpha = e^{\frac{1}{2}\alpha_i\phi^i}E_\alpha\,, \tag{2.18}$$

where $\mathbf{\Omega}$ is an $\dim\mathbf{n_k}\times\dim\mathbf{n_k}$ matrix

$$\begin{aligned}\mathbf{\Omega} &= \sum_{m=0}^{\infty}\frac{\omega^m}{(m+1)!}\\ &= (e^\omega - I)\,\omega^{-1}\,.\end{aligned} \tag{2.19}$$

The matrix ω is $\omega^\gamma_\beta = \chi^\alpha K^\gamma_{\alpha\beta}$. The notation we use for the commutation relations of the solvable Lie algebra $\mathbf{s}_0$ of $so(N,1)$ is as follows

$$[H_i, E_\alpha] = \alpha_i E_\alpha\,,\qquad [E_\alpha, E_\beta] = K^\gamma_{\alpha\beta}E_\gamma\,. \tag{2.20}$$

Since we have

$$[E_\alpha, E_\beta] = N_{\alpha,\beta}E_{\alpha+\beta}\,, \tag{2.21}$$

$K^\alpha_{\beta\beta}=0$ also $K^\alpha_{\beta\gamma}=N_{\beta,\gamma}$ if $\beta+\gamma=\alpha$ and $K^\alpha_{\beta\gamma}=0$ if $\beta+\gamma\neq\alpha$ in the root sense. By using (2.16) the last equation in (2.14) can also be written as

$$d(e^{-\sigma}L*F) = d\sigma\wedge e^{-\sigma}L*F - \mathcal{G}_0^T\wedge e^{-\sigma}L*F + 2(L^T)^{-1}\eta\, e^{-2\sigma}F\wedge e^{-\sigma}*G\,. \tag{2.22}$$

The field equations of the dilatons and the axions for a generic non-split scalar coset G/K coupled to matter fields have been derived in [9–11]. Thus referring to [7–11] by varying the lagrangian (2.11) with respect to the axions $\{\chi^m\}$ and the dilatons $\{\phi^i\}$ the corresponding field equations can be given as

$$\begin{aligned} d(e^{\frac{1}{2}\gamma_i\phi^i} * U^\gamma) &= -\frac{1}{2}\gamma_j e^{\frac{1}{2}\gamma_i\phi^i} d\phi^j \wedge *U^\gamma + \sum_{\alpha-\beta=-\gamma} e^{\frac{1}{2}\alpha_i\phi^i} e^{\frac{1}{2}\beta_i\phi^i} N_{\alpha,-\beta} U^\alpha \wedge *U^\beta \,, \\ d(*d\phi^i) &= \frac{1}{2}\sum_{\alpha\in\Delta^+_{nc}} \alpha_i e^{\frac{1}{2}\alpha_j\phi^j} U^\alpha \wedge e^{\frac{1}{2}\alpha_j\phi^j} * U^\alpha + 2e^{-2\sigma}((H_i)^A_N L^N_M L^A_J) * F^M \wedge F^J \,. \end{aligned} \tag{2.23}$$

Here the matrices $\{(H_i)^A_N\}$ are the matrix representatives of the generators $\{H_i\}$ in the $(N+1)$-dimensional representation we assume.

3. Dualisation and the first-order formulation

In this section we will apply the dualisation method of [5] to construct the coset formulation of the bosonic sector of the $D = 9$ matter coupled supergravity. We will also obtain the first-order bosonic field equations as a consequence of the coset formulation. Our formulation will be in parallel with the ones in [12, 13]. We will derive the Lie superalgebra which parameterizes the doubled or the dualized coset element so that the doubled Cartan form induced by it realizes the second-order field equations (2.14) and (2.23) by satisfying the Cartan-Maurer equation. Likewise in [5, 12, 13] the doubled Cartan form of the dualized coset element will obey a twisted self-duality equation which results in the first-order bosonic field equations of the theory. We start by assigning a generator for each bosonic field. The original generators $\{K, V_I, Y, H_j, E_m\}$ will be coupled to the fields $\{\sigma, A^I, B, \phi^j, \chi^m\}$ in the dualized coset parametrization. We will also introduce a dual field for each original field. The dual fields are $\{\widetilde{\sigma}, \widetilde{A}^I, \widetilde{B}, \widetilde{\phi}^j, \widetilde{\chi}^m\}$ where $\widetilde{B}$ is a five-form, $\{\widetilde{A}^I\}$ are six-forms and $\{\widetilde{\sigma}, \widetilde{\phi}^j, \widetilde{\chi}^m\}$ are seven-forms. These dual fields are the Langrange multipliers corresponding to the Bianchi identities of the relative original fields [6]. For example $\widetilde{B}$ is the Langrange multiplier which is coupled to the Bianchi identity of B in the construction of the Bianchi lagrangian term which leads to the first-order algebraic equations of motion etc. We will also assign the dual generators $\{\widetilde{K}, \widetilde{V}_I, \widetilde{Y}, \widetilde{H}_j, \widetilde{E}_m\}$ to the dual fields $\{\widetilde{\sigma}, \widetilde{A}^I, \widetilde{B}, \widetilde{\phi}^j, \widetilde{\chi}^m\}$ so that they will couple with each other in the parametrization of the dualized coset element. The Lie superalgebra of the original and the dual generators will have the $\mathbb{Z}_2$ grading so that the generators will be odd if the corresponding potential is an odd degree differential form and otherwise even [5]. The generators $\{V_I, \widetilde{K}, \widetilde{Y}, \widetilde{H}_j, \widetilde{E}_m\}$ are odd generators and $\{K, \widetilde{V}_I, Y, H_j, E_m\}$ are even ones. Specifically the doubled coset element will be parameterized by a differential graded algebra. This algebra is generated by the differential forms and the generators we have introduced above. The odd (even) generators behave like odd (even) degree differential forms under this graded differential algebra structure when they commute with the differential forms. The odd generators obey the anti-commutation relations while the even ones and the mixed couples obey the commutation relations.

To begin the construction of the coset formulation or the non-linear realisation of the bosonic sector of the $D = 9$ matter coupled supergravity we propose the dualized coset element

$$\nu' = e^{\frac{1}{2}\phi^j H_j} e^{\chi^m E_m} e^{\sigma K} e^{A^I V_I} e^{\frac{1}{2}BY} e^{\frac{1}{2}\widetilde{B}\widetilde{Y}} e^{\widetilde{A}^I \widetilde{V}_I} e^{\widetilde{\sigma}\widetilde{K}} e^{\widetilde{\chi}^m \widetilde{E}_m} e^{\frac{1}{2}\widetilde{\phi}^j \widetilde{H}_j} . \tag{3.1}$$

Here ν' is a map from the nine-dimensional spacetime into a group which is presumably the rigid symmetry group of the dualized lagrangian. Likewise in [5, 12, 13] we are not interested in the group theoretical structure of the non-linear realisation of the $D = 9$ matter coupled supergravity but rather in the Lie superalgebra which generates (3.1). Eventually this algebra also contains the information of the group theoretical structure of the coset formulation [7 − 10, 15, 16]. The local map ν' induces the doubled Cartan form $\mathcal{G}'$ on the nine-dimensional spacetime which can be given as

$$\mathcal{G}' = d\nu' \nu'^{-1} . \tag{3.2}$$

From its construction the Cartan form (3.2) satisfies the Cartan-Maurer equation

$$d\mathcal{G}' - \mathcal{G}' \wedge \mathcal{G}' = 0 . \tag{3.3}$$

The standard dualisation procedure [5, 7, 8, 11 − 13] requires the construction of the Lie superalgebra of the original and the dual generators such that they will lead us to the second-order equations of motion (2.14) and (2.23) when the Cartan form (3.2) is calculated and inserted in (3.3). Therefore by using the matrix formulas

$$\begin{aligned} de^X e^{-X} &= dX + \frac{1}{2!}[X, dX] + \frac{1}{3!}[X, [X, dX]] + \cdots , \\ e^X Y e^{-X} &= Y + [X, Y] + \frac{1}{2!}[X, [X, Y]] + \cdots , \end{aligned} \tag{3.4}$$

repeatedly we can calculate the Cartan form (3.2) in terms of the desired unknown structure constants of the Lie superalgebra of the original and the dual generators we have introduced. If we use this calculated Cartan form in the Cartan-Maurer equation (3.3) and then compare the result with the second-order bosonic equations (2.14) and (2.23) of the $D = 9$ matter coupled supergravity we can read the unknown structure constants and determine the Lie superalgebra structure which leads to the coset formulation of the $D = 9$ matter coupled supergravity. We will not present the details of this calculation here, similar calculations can be referred in [11 − 13]. However we should mention one point. Likewise in [12, 13] the scalar sector which is coupled to the matter fields in (2.11) can be treated separately on its own respect and one can make use of the general results obtained in [8, 11] to derive the commutation relations of the scalar generators and their duals. Therefore consequently if we perform the above mentioned calculation the comparison of (3.3) with the second-order bosonic field equations (2.14) and (2.23) leads us to the algebra structure whose commutation and the anti-commutation relations can be given as

$$\begin{aligned} &[K, V_I] = -V_I , && [K, Y] = -2Y , && [K, \widetilde{Y}] = 2\widetilde{Y} , \\ &[\widetilde{V}_I, K] = -\widetilde{V}_I , && \{V_I, V_J\} = \eta_{IJ} Y , && [H_l, V_I] = (H_l)_I^K V_K , \end{aligned}$$

$$
\begin{aligned}
&[E_m, V_I] = (E_m)^J_I V_J\,, \qquad [V_L, \widetilde{V}_M] = -\frac{2}{7}\delta_{LM}\widetilde{K} - \sum_i (H_i)_{LM}\widetilde{H}_i\,,\\
&\{V_K, \widetilde{Y}\} = 4\eta^L_K\,\widetilde{V}_L\,, \qquad [Y, \widetilde{Y}] = -\frac{16}{7}\widetilde{K}\,, \qquad [H_i, \widetilde{V}_K] = -(H_i^T)^M_K\widetilde{V}_M\,,\\
&[E_\alpha, \widetilde{V}_K] = -(E_\alpha^T)^M_K\widetilde{V}_M\,,\\
&[H_j, E_\alpha] = \alpha_j E_\alpha\,, \qquad [E_\alpha, E_\beta] = N_{\alpha,\beta}E_{\alpha+\beta}\,,\\
&[H_j, \widetilde{E}_\alpha] = -\alpha_j\widetilde{E}_\alpha\,, \qquad [E_\alpha, \widetilde{E}_\alpha] = \frac{1}{4}\sum_{j=1}^r \alpha_j\widetilde{H}_j\,,\\
&[E_\alpha, \widetilde{E}_\beta] = N_{\alpha,-\beta}\widetilde{E}_\gamma, \qquad \alpha-\beta=-\gamma,\ \alpha\neq\beta\,.
\end{aligned}
\tag{3.5}
$$

Here $(E_\alpha^T)^M_K$ and $(H_i^T)^M_K$ are the components of the transpose of the matrix representatives of the generators E_α and H_i respectively. The commutators and the anti-commutators which are not listed in (3.5) vanish. The commutation relations of the scalar generators and their duals form up a subalgebra and they are already derived in [8] for a generic scalar coset. If O represents the set of the original generators and $\widetilde{D}$ the set of the dual generators we conclude that the algebra constructed in (3.5) obeys the general scheme

$$
\begin{aligned}
&[O, \widetilde{D}\} \subset \widetilde{D}\,, \qquad [O, O\} \subset O\,,\\
&[\widetilde{D}, \widetilde{D}\} = 0\,.
\end{aligned}
\tag{3.6}
$$

Now that we have obtained the commutation and the anti-commutation relations of the original and the dual generators we can calculate the Cartan form $\mathcal{G}' = d\nu'\nu'^{-1}$ explicitly. From the definition of ν' in (3.1) by using the identities (3.4) effectively together with the algebra structure given in (3.5) the calculation of the Cartan form $\mathcal{G}'$ yields

$$
\begin{aligned}
\mathcal{G}' &= d\nu'\nu'^{-1}\\
&= \frac{1}{2}d\phi^i H_i + e^{\frac{1}{2}\alpha_i\phi^i}U^\alpha E_\alpha + d\sigma K + e^{-\sigma}L^K_I dA^I V_K + \frac{1}{2}e^{-2\sigma}GY +\\
&\quad + \frac{1}{2}e^{2\sigma}d\widetilde{B}\widetilde{Y} + \left(-\frac{4}{7}B\wedge d\widetilde{B} + \frac{2}{7}A^J\wedge d\widetilde{A}^I\delta_{IJ} + d\widetilde{\sigma}\right)\widetilde{K} +\\
&\quad + \sum_{m=1}^r \big((e^{\mathbf{\Gamma}}e^{\mathbf{\Lambda}})^m_j S^j + (H_m)_{JI}A^J\wedge d\widetilde{A}^I + \eta^K_I(H_m)_{JK}A^J\wedge A^I\wedge d\widetilde{B}\big)\widetilde{H}_m +\\
&\quad + \sum_{\alpha\in\Delta^+_{nc}} (e^{\mathbf{\Gamma}}e^{\mathbf{\Lambda}})^{\alpha+r}_j S^j\widetilde{E}_\alpha +\\
&\quad + \big(e^\sigma((L^T)^{-1})^L_K d\widetilde{A}^K + 2e^\sigma((L^T)^{-1})^L_K\,\eta^K_I\,A^I\wedge d\widetilde{B}\big)\widetilde{V}_L\,,
\end{aligned}
\tag{3.7}
$$

where we have defined $r = \dim \mathbf{h}_k$ and also $S^i = \frac{1}{2}d\widetilde{\phi}^i$ for $i = 1,\ldots,r$ whereas $S^{\alpha+r} = d\widetilde{\chi}^\alpha$ for $\alpha = 1,\ldots,\dim\mathbf{n}_k$. We assume that we enumerate the roots $\alpha\in\Delta^+_{nc}$ as $\alpha = 1,\ldots,\dim\mathbf{n}_k$. The matrices $\mathbf{\Gamma}$ and $\mathbf{\Lambda}$ have already been derived and given in [7, 8, 11–13] for general scalar cosets whose definitions are also applicable to the scalar coset $\mathrm{SO}(N,1)/\mathrm{SO}(N)$ of the matter coupled $D = 9$ supergravity. We should remark that in our entire formulation we use the euclidean signature metric to raise and lower the indices when needed for notational purposes. Since the dualisation of the supergravity theories is another manifestation of the Langrange multiplier methods [6] the dualized formulation of the theory also yields the first-order field equations in terms of the dual fields [5, 11 − 13]. By obeying the twisted self-duality equation

$$*\mathcal{G}' = \mathcal{S}\mathcal{G}' \,, \tag{3.8}$$

the Cartan form $\mathcal{G}'$ calculated in (3.7) leads us to the first-order field equations which can be obtained by the local integration of the second-order equations of motion (2.14) and (2.23) [5]. Here $\mathcal{S}$ is a pseudo-involution of the Lie superalgebra generated by the original and the dual generators. In general $\mathcal{S}$ sends the original algebra generators to their duals and the dual ones to their original counterparts with a sign factor which is $(-1)^{p(D-p)+s}$ where p is the degree of the field strength of the corresponding field which the generator is coupled to and s is the signature of the spacetime metric [5]. For our case of the $D = 9$ matter coupled supergravity the signature of the spacetime is assumed to be $s = 8$ [14]. So the sign factor becomes $(-1)^{p(9-p)+8}$. Thus the action of the pseudo-involution $\mathcal{S}$ on the generators can be given as

$$\begin{aligned}
\mathcal{S}Y &= \widetilde{Y} \,, & \mathcal{S}\widetilde{Y} &= Y \,, & \mathcal{S}E_m &= \widetilde{E}_m \,, & \mathcal{S}\widetilde{E}_m &= E_m \,, \\
\mathcal{S}K &= \widetilde{K} \,, & \mathcal{S}\widetilde{K} &= K \,, & \mathcal{S}H_i &= \widetilde{H}_i \,, & \mathcal{S}\widetilde{H}_i &= H_i \,, \\
\mathcal{S}V_I &= \widetilde{V}_I \,, & \mathcal{S}\widetilde{V}_I &= V_I \,. & & & &
\end{aligned} \tag{3.9}$$

Equipped with the above definition of the pseudo-involution $\mathcal{S}$ if we apply the twisted self-duality condition (3.8) on the dualized Cartan form (3.7) we obtain the first-order field equations as

$$\begin{aligned}
\frac{1}{2} e^{-2\sigma} * G &= \frac{1}{2} e^{2\sigma} d\widetilde{B} \,, \\
e^{-\sigma} L^I_J * dA^J &= e^{\sigma}((L^T)^{-1})^I_J d\widetilde{A}^J + 2e^{\sigma}((L^T)^{-1})^I_J \, \eta^J_K \, A^K \wedge d\widetilde{B} \,, \\
*d\sigma &= d\widetilde{\sigma} - \frac{4}{7} B \wedge d\widetilde{B} + \frac{2}{7}\delta_{IJ} A^J \wedge d\widetilde{A}^I \,, \\
e^{\frac{1}{2}\alpha_i\phi^i}(\mathbf{\Omega})^{\alpha}_l * d\chi^l &= (e^{\mathbf{\Gamma}} e^{\mathbf{\Lambda}})^{\alpha+r}_j S^j \,, \\
*d\phi^m &= 2(e^{\mathbf{\Gamma}} e^{\mathbf{\Lambda}})^m_j S^j + 2(H_m)_{JI} A^J \wedge d\widetilde{A}^I + 2\eta^K_I (H_m)_{JK} A^J \wedge A^I \wedge d\widetilde{B} \,.
\end{aligned} \tag{3.10}$$

Taking the exterior derivative of the equations given in (3.10) yields us the second-order field equations (2.14) and (2.23) which are independent of the dual fields naturally. This calculation is a straightforward one however two remarks must be made: firstly one should make use of the identities (2.7) and (2.8) which the coset elements obey especially in the differentiation of the dilaton equation in (3.10) to obtain the second-order dilaton equation in (2.23), secondly the general results of [11] which presents the dualisation and the first-order formulation of the generic matter-coupled non-split scalar cosets may be adopted for the differentiation of the dilaton and the axion equations in (3.10) since these scalar fields parameterize the coset $\mathrm{SO}(N,1)/\,\mathrm{SO}(N)$ where $\mathrm{SO}(N,1)$ is a non-compact real form of a semi-simple Lie group and thus $\mathrm{SO}(N,1)/\,\mathrm{SO}(N)$ is in general a non-split scalar coset as we have mentioned before.

4. Conclusion

By introducing dual fields for the bosonic field content we have performed the method of dualisation [5] to obtain the non-linear realisation of the bosonic sector of the $D = 9$ matter coupled supergravity [14]. For the scalars of the coupling vector multiplets we have given the symmetric space sigma model formulation which is based on the $\mathrm{SO}(N,1)/\mathrm{SO}(N)$ coset manifold in section two. In section three we have extended the non-linear formulation of the scalars to the entire bosonic sector excluding the graviton. We have doubled the bosonic field content by defining dual fields. After introducing an algebra generator for each field we have proposed a doubled coset element. Our objective has been to construct the Lie superalgebra of the field generators which parameterize the doubled coset element. We have derived the necessary commutation and the anti-commutation relations of the generators by calculating the dualized Cartan form due to the dualized coset element in terms of the unknown structure constants and by inserting it into the Cartan-Maurer equation then by comparing the result with the second-order field equations which have been given in section two. In summary we have derived the Lie superalgebra which realizes the bosonic field equations of the $D = 9$ matter coupled supergravity by parameterizing a coset element whose Cartan form yields the field equations in the Cartan-Maurer equation. Thus this construction leads us to the bosonic non-linear realisation of the $D = 9$ matter coupled supergravity. We have also obtained the first-order bosonic field equations of the $D = 9$ matter coupled supergravity as a twisted self-duality equation [5] of the dualized Cartan form. Thus the dualisation method which is closely related to the Lagrange multiplier construction [6] gives us also the first-order equations of motion in terms of the dual fields.

In this work although we have derived the algebra structure which parameterizes the dualized coset element we have neither shown an attempt to identify the algebra nor worked on the group theoretical construction of the coset. One can focus on the structure of the algebra given in section three and inspect its correspondence with the ones formulated in [12] and [13]. We can also examine the local and the global symmetries of the first-order field equations and the dualized Cartan form and these symmetries can be linked to the coset parametrization we have constructed. This analysis would give the group theoretical construction of the dualized coset. The graviton can also be included in the non-linear coset formulation presented here. This would lead one to a non-linear realisation of the theory in terms of the Kac-Moody algebras [18 – 20]. One can extend the non-linear coset formulation to include the fermions as well. The knowledge of the global and the local symmetries which can be extracted from the coset formulation presented here will have a contribution to the understanding of the symmetries of the ten dimensional heterotic string theory since the $D = 9$ matter coupled supergravity can be obtained as a Kaluza-Klein descendant theory of the low energy limit of the ten dimensional heterotic string [21]. Such a symmetry inspection would be improved by the more general non-linear realisation of the matter coupled $D = 9$ supergravity which would include the gravity sector and whose formulation would be based on a Kac-Moody symmetry group as in [18 – 20].

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